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$W_{1+\infty}$ FIELD THEORIES FOR THE EDGE EXCITATIONS IN THE QUANTUM HALL EFFECT*

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We briefly review these low-energy effective theories for the quantum Hall effect, with emphasis and language familiar to field theorists. Two models have been proposed for describing the most stable Hall plateaus (the Jain series): the multi-component Abelian theories and the minimal $W_{1+\infty}$ models. They both lead to a-priori classifications of quantum Hall universality classes. Some experiments already confirmed the basic predictions common to both effective theories, while other experiments will soon pin down their detailed properties and differences. Based on the study of partition functions, we show that the Abelian theories are rational conformal field theories while the minimal $W_{1+\infty}$ models are not.

1. Introduction: the Incompressible Fluid, the Edge Excitations and the w_∞ Symmetry

In the quantum Hall effect[1], the *effective field theory* approach has been developed to a rather high degree of sophistication. Its basic hypothesis are indeed fulfilled: the values of the Hall conductivity σ_{xy} at the plateaus ($\sigma_{xx} = 0$) display *universality*; moreover, the nature of the ground state and of the low-energy excitations have been understood, together with their characteristic symmetry. Laughlin[2] made the fundamental observation that the electrons form an *incompressible fluid*: this means that the density is uniform inside the sample *(fluid character)* and that density waves have a gap *(incompressibility)*. Actually, the latter property implies the absence of longitudinal conduction.

Most of the properties of the incompressible fluid can be understood at the (semi)-classical level.[2 3] A *droplet* of classical incompressible fluid is defined by the

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density function,

$$\rho(z,\bar{z},t) = \rho_0\ \chi_{S_A(t)}\ , \qquad A = \frac{N}{\rho_0}\ , \tag{1}$$

where $\chi_{S_A(t)}$ is the characteristic function for a surface $S_A(t)$ of area A, N is the particle number, and $z = x + iy$, $\bar{z} = x - iy$ are complex coordinates on the plane. This description is valid for energies far below the gap for density waves, so that the density is uniform (ρ_0) and the area of the droplet is *constant*. A convenient measure of the density in the quantum mechanical problem is given by the *filling fraction* ν,

$$\nu = \frac{N}{\mathcal{D}_A}\ , \qquad \mathcal{D}_A = \frac{BA}{hc/e}\ ; \tag{2}$$

$\mathcal{D}_A$ is the number of one-particle states in a Landau level, which is given by the magnetic flux through the surface in units of the flux quantum $\phi_o = hc/e$. The simplest example of incompressible fluid is given by the completely filled first Landau level ($\nu = 1$), where the gap for density waves is of the order of the cyclotron energy $\omega_c = eB/mc$ separating two Landau levels.

The Hall conductivity is simply expressed in terms of the filling fraction

$$\sigma_{xy} = \frac{e^2}{h}\ \nu\ , \tag{3}$$

because the droplet drifts as a rigid body under the effect of an in-plane electric field. The experimental values[4] show a characteristic *hierarchical pattern* (see Figure one). For the filling fractions

$$\nu = \frac{m}{ms \pm 1}\ , \qquad m = 1, 2, \ldots, \qquad s = 2, 4, \ldots, \tag{4}$$

Jain[5] introduced a generalization of the original Laughlin trial wave functions which correctly matches the microscopic dynamics. Based on the physical picture of the *composite fermion* excitation, Jain was able to map strongly-interacting electrons at the filling fractions (4) into weakly-interacting composite fermions at the effective integer filling $\nu^* = m$. By carrying over the stability of completely filled Landau levels, he argued that the filling fractions (4) form one-parameter families of approximately equally stable Hall fluids, at fixed s, which accumulate at $\nu \to 1/s$. These are called the Jain series and are represented in Figure one by the **bold** fractions; the remaining points (*italic* fractions) are less understood. Here we shall not discuss the wave-function approach any further, but instead show that the Jain series can be independently derived within the effective field theory approach.

Since the droplets of incompressible fluid have constant area, they can only change their shape in response to external forces. The infinitesimal deformations are the *edge excitations* [6], which are $(1+1)$-dimensional *chiral* waves. Another type of excitations are the classical *vortices* in the bulk of the droplet, which are localized holes or dips in the density. The absence of density waves implies that any density excess or defect is completely transmitted to the boundary, where it is seen as a

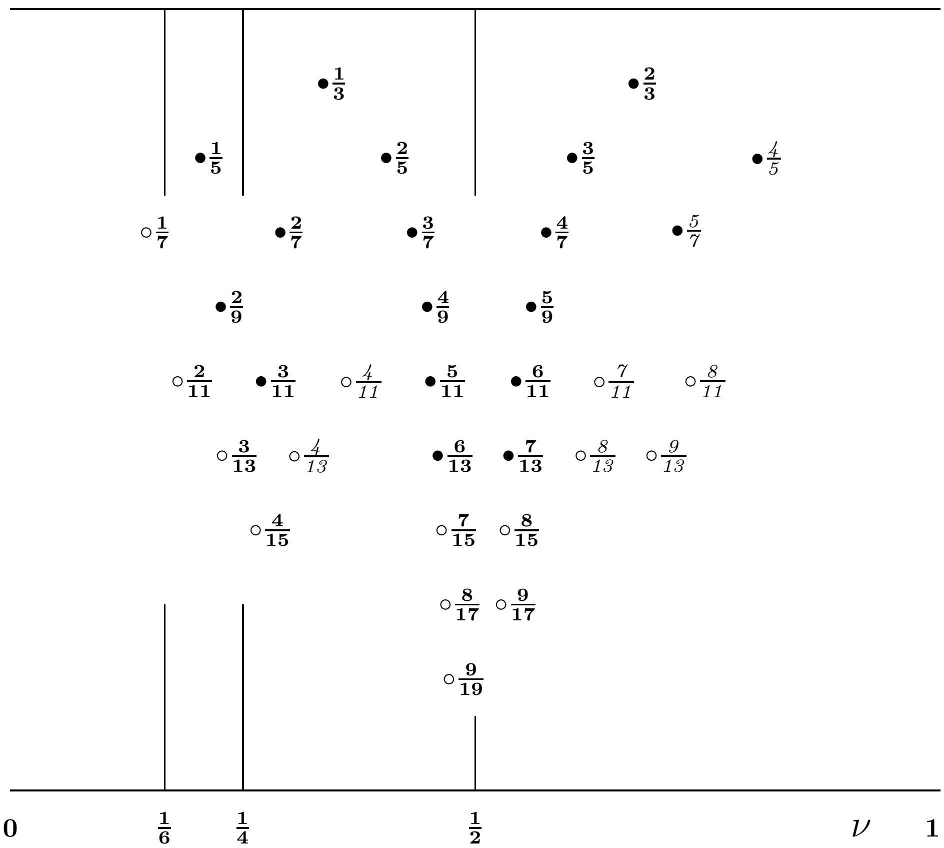

Fig. 1. Experimentally observed plateaus: their Hall conductivity is displayed in units of (e^2/h). The marks (•) denote stable (i.e. large) plateaus, which have been seen in several experiments; the marks (∘) denote less developed plateaus and plateaus found in one experiment only. Note that the stability decreases as the denominator of ν increases. Coexisting fluids at the same filling fraction have been found at $\nu = 2/3, 2/5, 3/5, 5/7$.

further edge deformation. The matter displaced at the edge defines the *charge* of the vortices; instead, the edge waves are neutral excitations.

The classical configuration space of a droplet of incompressible fluid, describing edge waves and vortices, is completely spanned by the generators of an infinite-dimensional algebra, the w_∞ algebra of *area-preserving diffeomorphisms* of the two-dimensional plane. This is the w_∞ *dynamical symmetry* of two-dimensional incompressible fluids.[7] [8] In order to describe this symmetry,[3] let us consider the ground state density in a rotation-invariant potential,

$$\rho_{GS}(z, \bar{z}) = \rho_0 \ \Theta\left(R^2 - z\bar{z}\right) \ . \tag{5}$$

The small deformations of the droplet at constant area can be generated by reparametrizations of the coordinates of the plane with unit Jacobian, the area-preserving

diffeomorphisms. A familiar example of these reparametrizations is given by the canonical transformations of a two-dimensional *phase space*. If we identify the $(z, \bar{z})$ coordinate plane with a phase space, we can describe the area-preserving diffeomorphisms in terms of canonical transformations. Define the dimensionless Poisson brackets $\{f, g\} \equiv i\ \left(\partial f \bar{\partial} g - \bar{\partial} f \partial g\right)/\rho_0$, where $\partial \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial/\partial \bar{z}$. Thus, the area-preserving diffeomorphisms are given by $\delta z\ =\ \{\mathcal{L}^{(cl)}, z\}$ and $\delta \bar{z}\ =\ \{\mathcal{L}^{(cl)}, \bar{z}\}$, in terms of the generating function $\mathcal{L}^{(cl)}(z, \bar{z})$ of both "coordinate" and "momentum". The basis of generators $\mathcal{L}^{(cl)}_{n,m} \equiv \rho_0^{n+m/2}\ z^n \bar{z}^m$ satisfy the w_∞ algebra[8],

$$\left\{\mathcal{L}^{(cl)}_{n,m}, \mathcal{L}^{(cl)}_{k,l}\right\} = -i\ (mk - nl)\ \mathcal{L}^{(cl)}_{n+k-1,m+l-1}\ . \tag{6}$$

The *small* excitations above the ground state are thus given by the infinitesimal w_∞ transformations of ρ_{GS} in (1), namely by $\delta\rho_{n,m} \equiv \left\{\mathcal{L}^{(cl)}_{n,m}, \rho_{GS}\right\}$. By using the Poisson brackets, one finds that $\delta\rho_{n,m}$ is indeed localized at the edge and that it describes a chiral wave of momentum $k = n - m$ (the Fourier mode on the circle). We can also consider other "ground state" droplets with a given vorticity in the bulk, and then construct the corresponding basis of edge waves. Thus, the whole configuration space of the excitations of a classical incompressible fluid is spanned by infinitesimal w_∞ transformations, as we anticipated.

The classical edge waves and vortices have quantum analogues in the Laughlin theory[2], which describes the filling fractions $\nu\ =\ 1, 1/3, 1/5, \ldots$. In the simplest case of $\nu\ =\ 1$, the droplet of electron fluid is actually a filled Fermi sea[9]: the edge waves are particle-hole excitations across the Fermi surface represented by the edge of the droplet, and the vortices correspond to localized quasi-particle and quasi-hole excitations in the bulk of the fluid. For fractional fillings, the quantum incompressible fluid is not easy to understand at the microscopic level: nevertheless, Laughlin[2] showed that the quasi-particles possess *fractional* charge and statistics.

In the quantum theory, there is also a relation[8] between the edge excitations and the generators of the quantum version of w_∞, called $W_{1+\infty}$. This algebra is obtained by replacing the Poisson brackets with quantum commutators: $i\{\ ,\ \}\ \rightarrow\ [\ ,\ ]$, and by taking the thermodynamic limit of large droplets[10]. In this limit, the radius of the droplet grows as $R \propto \ell\sqrt{N} \rightarrow \infty$, where $\ell = \sqrt{2\hbar c/eB}$ is the magnetic length. Quantum edge excitations, instead, are confined to a boundary annulus of finite size $O(\ell)$. Therefore, for $\nu = 1$, the Fermi sea can be approximated by the relativistic Dirac sea of a Weyl fermion.[10] The $W_{1+\infty}$ generators can be obtained canonically in this theory, and satisfy the $W_{1+\infty}$ algebra,[11]

$$\left[\ V^i_n, V^j_m\ \right] = (jn - im)\ V^{i+j-1}_{n+m} + q(i,j,m,n)\ V^{i+j-3}_{n+m} + \ldots + c\ d^i(n)\ \delta^{i,j}\delta_{n+m,0}\ . \tag{7}$$

In this equation, $i+1 \geq 1$ is the "conformal spin" of the generator V^i_n and $-\infty < n < +\infty$ is the edge momentum. The first term on the right-hand-side of (7) reproduces the classical w_∞ algebra (6) by the correspondence $\mathcal{L}^{(cl)}_{i-n,i} \rightarrow V^i_n$ and identifies $W_{1+\infty}$ as the algebra of "quantum area-preserving diffeomorphisms". Moreover, there are some quantum corrections with polynomial coefficients $q(i,j,n,m)$, and

the c-number term $c\ d^i(n)$ due to the (relativistic) quantum anomaly.[12] All terms in the commutation relations (7) are known and uniquely determined by the closure of the algebra; only the *central charge* c is a free parameter ($c = 1$ for the Weyl fermion). The generators V^0_n are Fourier modes of the fermion density evaluated at the edge $|z| = R$, thus V^0_0 measures the edge charge; instead, V^1_0 measures the angular momentum of edge excitations.[10] These generators obey the Abelian current algebra $\widehat{U(1)}$ and the Virasoro algebra, respectively[12], which are important sub-algebras of the $W_{1+\infty}$ algebra.

The presence of a dynamical symmetry allows for a more abstract description of the Weyl fermion theory:[13] instead of writing the Hamiltonian and then proceed via canonical quantization, we can obtain the Hilbert space by assembling the set of irreducible, highest-weight representations of the $W_{1+\infty}$ algebra, which is closed under the *fusion rules* for making composite excitations (this is the general procedure for building conformal field theories[12]). Any $W_{1+\infty}$ representation contains a bottom state, the *highest weight* state $|Q\rangle$, and an infinite tower of particle-hole excitations above it. The bottom state represents the droplet ground-state (5) ($Q = 0$), which might have quasi-particles in it ($Q \neq 0$). The charge Q and the spin J of the quasi-particle are given by the eigenvalues $V^0_0 \mid Q \rangle = Q \mid Q \rangle$ and $V^1_0 \mid Q \rangle = J \mid Q \rangle$, while the fractional statistics θ/π is twice the spin J.

2. Classification of QHE Universality Classes by the $W_{1+\infty}$ Symmetry: Minimal and Non-Minimal Models

We have seen that the w_∞ dynamical symmetry describes the geometry of classical incompressible fluids and that, for $\nu = c = 1$, the Hilbert space for quantum edge excitations is similarly made of $W_{1+\infty}$ representations. Actually, the effective quantum field theories of incompressible fluids are characterized by the $W_{1+\infty}$ symmetry for general filling fractions, because there is a unique quantization of the w_∞ algebra in $(1 + 1)$-dimensional field theory.[11] We can therefore *postulate* that *all* universality classes of quantum Hall incompressible fluids are in one-to-one correspondence with $W_{1+\infty}$ theories.[13] These classes are specified by the *kinematical data* of the charges Q and fractional statistics θ/π of the quasi-particles (the eigenvalues of V^0_0 and V^1_0). Moreover, the Hall conductivity (3) can be obtained from the chiral anomaly of the $(1 + 1)$-dimensional theory.[10 13] One can also consider other quantities, like the characters of the $W_{1+\infty}$ algebra: they give the number of particle-hole excitations above the ground state, which can be checked in numerical simulations of the electron fluid.[6]

This classification program can be carried through because all $W_{1+\infty}$ unitary, irreducible, highest-weight representations have been found.[11] They exist for positive integer central charge $c = m = 1, 2, \ldots$ and are labeled by a m-component highest-weight vector $\vec{r} = \{r_1, \ldots, r_m\}$, whose sum of components gives the charge Q. The results of the representation theory can be summarized as follows:

- If $c = 1$, the $W_{1+\infty}$ representations are completely equivalent to those of its

Abelian sub-algebra $\widehat{U(1)}$.

- If $c = 2, 3, \ldots$ there are two kinds of representations, *generic* and *degenerate*, depending on the type of weight. The generic representations occur for $(r_i - r_j) \notin \mathbf{Z}$, $\forall\ i \neq j$, and are actually *equivalent* to the corresponding representations of the multi-component Abelian algebra $\widehat{U(1)}^{\otimes m}$ having the same weight.
- The degenerate representations have weights with some $(r_i - r_j) \in \mathbf{Z}$. These representations are *contained* in the corresponding $\widehat{U(1)}^{\otimes m}$ representations, i.e. the latter are reducible $W_{1+\infty}$ representations. The degenerate $W_{1+\infty}$ representations can be obtained from the $\widehat{U(1)}^{\otimes m}$ ones by projecting out an infinity of states in the tower of excitations.

It is rather amusing to observe that this table of mathematical results matches the list of known effective field theories for the quantum Hall effect, which have been proposed by several group and were often based on different hypotheses:

- The $c = 1$ Abelian theory describes the Laughlin fluids with spectrum

$$\nu = \frac{1}{p}\ , \quad Q = \frac{n}{p}\ , \quad J = \frac{n^2}{2p}\ , \qquad n \in \mathbf{Z}\ , \quad p = 1, 3, 5, \ldots\ , \tag{8}$$

  where n is the number of quasi-particles. This effective field theory is consistent with the original Laughlin wave-function approach and is well established.[6] The spectrum (8) and the number of edge excitations above the ground state have been confirmed by numerical and real experiments (more on this later). An explicit action for this theory is given by the $(1 + 1)$-dimensional chiral boson (also called chiral Luttinger liquid). This theory is equivalent, for $\nu = 1$, to the Weyl fermion theory by bosonization, and, in general, to the Abelian Chern-Simons topological gauge theory defined on the two-dimensional plane.[14]
- The multi-component generalization of the Abelian theory,[14] [15] carrying the $\widehat{U(1)}^{\otimes m}$ symmetry, is the "standard model" for generic filling fractions. Its spectrum is given by

$$Q = \sum_{i,j=1}^{m}\ K^{-1}_{ij}\ n_j\ , \qquad \frac{\theta}{\pi} = \sum_{i,j=1}^{m}\ n_i\ K^{-1}_{ij}\ n_j\ , \qquad \nu = \sum_{i,j=1}^{m}\ K^{-1}_{ij}\ . \tag{9}$$

  Besides the central charge $c = m$, this theory is specified by a $(m \times m)$ symmetric, integer-valued matrix K_{ij} of couplings (with K_{ii} odd). The $\widehat{U(1)}^{\otimes m}$ representations in this spectrum correspond to $c = m$ $W_{1+\infty}$ representations whose weights $\vec{r}_i$ are given by the "metric" $K^{-1}_{ij} \sim \vec{r}_i \cdot \vec{r}_j$. Therefore, generic forms of K correspond to generic $W_{1+\infty}$ representations.
- However, it turns out that the relevant Jain filling fractions (4) are described by the specific matrices $K = 1 + s\ C$, where $C_{ij} = 1$, $\forall i, j = 1, \ldots, m$. These

theories are characterized[14 15] by the extended symmetry $\widehat{U(1)} \otimes \widehat{SU(m)}_1 \supset \widehat{U(1)}^{\otimes m}$ and, moreover, are made of those Abelian representations which are *inequivalent* to the (degenerate) $W_{1+\infty}$ representations.[16]

- Therefore, for the Jain filling fractions, *two* different kinds of $W_{1+\infty}$ theories can be built, the *non-minimal* and *minimal* ones:[16] the Abelian theories with $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ symmetry are made of reducible $W_{1+\infty}$ representations and are thus non-minimal; the $W_{1+\infty}$ minimal models are made of irreducible degenerate representations. Note that all the $W_{1+\infty}$ minimal models were independently built in Ref.16 and were shown to exist for the Jain filling fractions *only*.

In conclusion, there are two effective field theories describing the Jain series: the Abelian theories with enhanced $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ symmetry and the $W_{1+\infty}$ minimal models. Both theories display the same spectrum of charge and fractional spin of quasi-particles, but have different multiplicities of excitations. The $W_{1+\infty}$ minimal models are equivalent to the $\widehat{U(1)} \otimes \mathcal{W}_m$ conformal theories[11], where $\mathcal{W}_m$ is the Zamolodchikov-Fateev-Lykyanov algebra.[17] Since the $\mathcal{W}_m$ representations are isomorphic to those of the $SU(m)$ Lie algebra, the neutral excitations of the minimal models have associated an $SU(m)$ *"isospin"* quantum number, i.e. they are *quark-like* and their statistics is *non-Abelian*.[16] This means that the four-point functions have several intermediate channels, which can be observed in the scattering of two quasi-particles. Moreover, the number of excitations above the ground state is smaller in the $W_{1+\infty}$ minimal models than in the Abelian ones, due to the corresponding inclusion of representations. A more detailed analysis[18] shows that the minimal models have $SU(m)$ quantum numbers but cannot realize the full $SU(m)$ symmetry (for ex., in the $SU(2)$ case, J_+ is present, while J_-, J_0 are missing).

Present theoretical a-priori arguments, like consistency conditions or symmetries cannot choose between one of the two theories. In the literature, the non-minimal $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ theory was first introduced and is more widely accepted. It has been argued that the $\nu = m$ plateaus possess m independent edges (m Weyl fermions), which naturally realize the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ symmetry: this might extend to the Jain plateaus by the composite-fermion correspondence. However, the m Landau levels are not equivalent at the microscopic level, as well as in the composite-fermion theory, even for $\omega_c = 0$. Unfortunately, a direct relation between the Jain theory and the edge approach has not been found yet. Finally, the numerical experiments counting the number of excited states could distinguish between the two theories, but they are not accurate enough at present.

3. Experiments

Let us now review the experiments which have already confirmed the simpler $c = 1$ Abelian theory of edge excitations and discuss further experiments which could identify one of the two proposed $c > 1$ theories:

- The "time domain" experiment[19] has observed the propagation of waves along the edge of a disk sample; a short pulse was injected at one point on the edge and detected at another edge point by a fast oscilloscope. This confirmed that the $\nu = 1/3$ Hall fluid has a single chiral excitation as well as the $\nu = 2/3$ one (which has a second neutral excitation according to both effective theories)
- The characteristic low-energy spectrum of edge excitations $\epsilon_k \sim (k \log k)/R$ was found by an experiment of *radio-frequency* resonance.[20] This was not considered a proof of the edge states because the spatial propagation could not be resolved.
- An interesting recent experiment is the resonant tunneling through a point contact, proposed in Ref.21 and first done in Ref.22. Usually, edge excitations are chiral and do not self-interact; however, one can consider a rectangular geometry and pinch the electron fluid at the mid point by applying a localized potential barrier, such that the two opposite edges interact. One observes resonance peaks in the conductivity when a quasi-particle or an electron tunnels through the point contact. At $\nu = 1/3$, the Abelian theory predicts the anomalous temperature dependence $T^{2/3}$ (as compared to the free Fermi liquid) of the half-width of the resonance peaks and a characteristic peak shape.[21] The first observation of these effects has not been confirmed by other experiments and the current belief is that they should be seen at much lower temperatures.[23] Nevertheless, another tunneling experiment has measured the fractional charge of the quasi-particles.[24]

Generalizations of the tunneling experiments to other filling fractions $\nu = 2/3, 2/5, \ldots$ will test the $c > 1$ effective theories; however, their characteristic neutral excitations have indirect effects in conduction experiments. Moreover, an interference experiment involving the four-point function must be devised to distinguish between the Abelian and non-Abelian statistics which characterize the two proposed theories. Perhaps, this could be achieved in the tunneling experiment with two point contacts recently proposed in Ref.25. In conclusion, we hope to have convinced the reader that the effective theories of the quantum Hall effect are now being tested by very interesting experiments.

4. The Abelian Theories are Rational Conformal Field Theories

The quantum numbers of edge excitations always take rational values; this suggests that the corresponding conformal field theories should belong to the special, well understood class of *rational* conformal field theories (RCFT)[26]. By definition, these contain a *finite* number of representations of an extended (chiral) symmetry algebra, which includes Virasoro as a sub-algebra: clearly, each of the representations describes a sector of the Hilbert space which contains infinite states. These representations are encoded in the partition function[12] of the Euclidean theory defined

on the space-time torus $S^1 \times S^1$,

$$Z(\tau) = \sum_{\lambda,\bar{\lambda}=1}^{N} \mathcal{N}_{\lambda\bar{\lambda}}\ \chi(\tau)_\lambda\ \overline{\chi(\tau)}_{\bar{\lambda}}\ , \tag{10}$$

where τ is the ratio of the two periods of the torus, χ_λ are the characters of the extended algebra representations and $\mathcal{N}_{\lambda\bar{\lambda}}$ are the (unknown) multiplicities of the representations. The partition function gives a precise inventory of all the states in the Hilbert space and serves as a *definition* of the RCFT. In the literature,[12] it was shown that Z is invariant under the modular transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$, which span the discrete group $\Gamma = SL(2,\mathbf{Z})/\mathbf{Z}_2$. Furthermore, Verlinde[27] has shown the relation between modular invariance and the fusion rules of the extended symmetry algebra. Moreover, Witten[28] has explained that any RCFT is associated to a Chern-Simons theory, and that the torus partition function in the former theory corresponds to a path-integral amplitude for the latter theory on the manifold $S^1 \times S^1 \times \mathbf{R}$, where $\mathbf{R}$ is the time axis.

In the quantum Hall effect, a complete description of the edge excitations similarly requires the construction of their partition function.[18] Consider a spatial annulus with Euclidean compact time: this space-time manifold is topologically $\mathcal{M} = S^1 \times S^1 \times I$, because the radial coordinate runs over the finite interval I and the angular coordinate and Euclidean time are both compact. We can define the partition function of the conformal field theory on the edge(s) $\partial\mathcal{M} = S^1 \times S^1$, which is a space-time torus. This annulus partition function takes the standard form (10), where the characters of the chiral and antichiral algebras pertain to the inner and outer edges, respectively.

The construction of the partition functions for the simpler $c = 1$ Abelian theories shows that these are indeed rational conformal field theories; for $\nu = 1/p$, the extended symmetry algebra is the extension of $\widehat{U(1)}$ by a current of Virasoro dimension $h = p/2$, and the Verlinde fusion rules are the Abelian group $\mathbf{Z}_p$ (addition modulo p). The number p of extended algebra representations is equal to the Wen *topological order*,[29] which is the degeneracy of the quantum Hall ground state on the (ideal) compact surface of the torus - another universal property of Hall fluids. This degeneracy is usually accounted for by the effective Chern-Simons theory, but it can be equivalently obtained from the annulus partition function, thanks to the Verlinde and Witten relations.[18]

The partition functions for the two $c > 1$ effective theories show new features:[18]

- For the Abelian $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ theories, there are several solutions to the modular invariance conditions, leading to chiral-antichiral *diagonal* and *non-diagonal* partition functions.
- The partition functions corresponding to the minimal $W_{1+\infty}$ theories cannot be modular invariant.

Let us first discuss the Abelian theories. A diagonal partition function ($\mathcal{N}_{\lambda\bar{\lambda}} = \delta_{\lambda\bar{\lambda}}$ in Eq.(10)) is found for each of the *chiral* theories previously discussed. It

contains the excitations in Eq.(9) for each edge of the annulus, and, moreover, paired two-edge excitations. On the other hand, the non-diagonal partition functions define new RCFTs, which describe further observed plateaus beyond the Jain series, as follows. They exist for m containing a square factor, $m = a^2 m'$, and possess the extended symmetry algebra $\widehat{U(1)} \otimes \widehat{SU(a^2 m')}_1$ and a smaller set of neutral quasi-particles. Their filling fractions span again the Jain series, with $m \to m'$, and further series of fractions which include $\nu = 2/3, 4/5, 6/7, 8/9, \ldots, 4/11, 4/13$ (only) for small denominator. The experimental points beyond the Jain series (in italics in Figure one) match rather well these new values and their (less stable) "charge-conjugated" partners $\nu \to (1-\nu)$. This has to be compared with the higher orders of the Jain hierarchy of wave functions[5] and similar constructions, which generically predict too many unobserved filling fractions.

Next, we discuss the $W_{1+\infty}$ minimal models. Since they do not have a modular invariant partition function, they are not rational conformal field theories and do not correspond to Chern-Simons theories. Therefore, they are less understood in the literature. Clearly, they are fully consistent conformal field theories and their relevance is mainly an experimental question, as discussed before. Presumably, the projection from the Abelian to the minimal theories (the Hamiltonian reduction[30]) is incompatible with modular invariance. Their partition functions should be found by a novel approach, in which modular invariance would be appropriately generalized, possibly along the lines of Ref.31.

In conclusions, we would like to mention the model building associated with other quantum Hall states with spinful or layered electrons[32] (called *non-Abelian* Hall fluids for historical reasons). The simple semiclassical droplet model of section one does not apply to these more involved fluids; thus, their effective edge theories might fall outside the $W_{1+\infty}$ classification. We remark that the modular invariant partition function and the associated RCFT calculus can be useful for understanding these edge theories, as initiated in Ref.33.

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